Characterization of the Impact of Power Normalization on the MIMO Mutual Information

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Abstract— **In this paper, we investigate the impact of power normalization on the mutual information of Multiple-Input Multiple-Output (MIMO) systems. We quantify the difference in the mutual information when using two different power normalization methods - corresponding to slow and fast power control - and characterize this difference in the high SNR regime in terms of its distribution, mean and variance. We assume Rayleigh fading channels with perfect channel state information at the receiver. Also, we assume equal power allocation strategy is applied at the transmitter. It turns out that the difference in mutual information is governed by the temporal variations of the SNR. In MIMO systems, the inherent spatial diversity of the channel makes the difference negligible in Rayleigh fading environments.**

I. INTRODUCTION

Increasing demand for capacity in recently emerging high bit rate applications such as wireless internet and mobile multimedia services have continuously motivated researchers to develop new architectures supporting higher data rates. Since the pioneering works of Foschini [1] and Telatar [2] in showing potentially high spectral efficiency for Multiple-Input Multiple-Output (MIMO) systems, lots of efforts has been put into exploring fundamental limits of these systems and developing methods to capitalize on this potential [3]–[5]. In parallel, many measurement campaigns have been conducted to verify the performance of MIMO communications in real-life. However, experimental data collected in measurements is subject to non-ideality of instruments and undesired effects of the propagation environment (see [6] and [7] for further information.) Therefore, some sort of data processing is crucial to exclude effects of instrument impairments and undesired propagation phenomena from the measurements.

As part of data processing, power normalization is usually considered in experimental data analysis and channel modeling. Such normalization aims at removing the effect of average path loss (large-scale fading) from measurement data or the model. Thus, the capacity becomes independent of the transmitter-receiver separation and their locations as long as the environment statistics remain unchanged. Hence, normalization is of prime importance in correct interpretation of the results.

Two kinds of normalization are usually employed by researchers [7]–[9]. In first, each measured channel matrix is normalized to keep its Frobenius norm constant. In the second method however, all channel snapshots (or realizations) are multiplied by a constant normalization factor such that the average of their Frobenius norm becomes fixed. As described in the next section, the first normalization keeps the instantaneous SNR fixed at the receiver whereas the second one makes the average SNR fixed. It is noted that from a system point of view, these two normalization methods simulate the function of fast (perfect) and slow (imperfect) power control, respectively.

In this paper, we investigate the impact of the two discussed normalization methods (and thus the power control) on the mutual information (MI) of MIMO communication systems.

The rest of this paper is organized as follows: Section II introduces the system model, temporal SNR and the two power normalization methods. Section III derives the difference in the MI when using the two normalization methods, based on a high SNR approximation of the MI. Section IV provides statistical analysis of the difference. A discussion on the results is presented in Section V and finally Section VI concludes the paper.

Notations: Throughout the paper, boldface lower case and upper case letters are used to indicate vectors and matrices respectively. $[\mathbf{H}]_{ij}$ is used to denote the element in the i^{th} row and j^{th} column of the matrix $\mathbf{H}$. $\|\mathbf{H}\|_F$, $|\mathbf{H}|$ and $\mathbf{H}^*$ stand for the Frobenius norm, determinant and conjugate transpose of $\mathbf{H}$, respectively. $\mathrm{E}[x]$ and $\mathrm{var}[x]$ denote expectation and variance of random variable x . $x \sim \mathcal{CN}(0,1)$ indicates that x has a complex Gaussian random distribution with zero mean and unit variance.

II. SYSTEM MODEL

Consider a MIMO channel with N_t antennas at the transmitter and N_r antennas at the receiver. The channel input-output relationship is given by

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$$\mathbf{y} = \sqrt{\frac{P}{N_t}}\mathbf{Hx} + \mathbf{n}. \tag{1}$$

where $\mathbf{y}$ and $\mathbf{x}$ are output and input vectors respectively and $\mathbf{n}$ is the vector of independent and identically distributed (i.i.d.) noise samples of variance ${\sigma_n}^2$ at the receiver. P is the total transmit power and $\mathbf{H}$ represents the $N_r \times N_t$ channel matrix. For simplicity of notation, we define $K = \min(N_t, N_r)$ and $L = \max(N_t, N_r)$.

From (1) it is easily followed that the instantaneous SNR per receive antenna (averaged over all receive antennas) is given by

$$\rho = \frac{P}{{\sigma_n}^2}\frac{\|\mathbf{H}\|_F^2}{N_t N_r}. \tag{2}$$

To ensure a fair comparison of the capacity performance for different MIMO systems and configurations, two conditions are usually considered: the same transmit power and the same receive SNR. The first condition ensures that the total transmit power will be the same for all systems and configurations in the study, regardless of the number of antennas and the power allocation strategy. While here, we have fixed the total power as P , channel matrix normalization is essential in meeting the second condition- the same receive SNR- as follows:

Considering $\{\mathbf{H}_i\}_{i=1}^{N_s}$ as a sequence of N_s successively measured channel matrices, $\tilde{\mathbf{H}}_i$ and $\hat{\mathbf{H}}_i$ represent two normalized channel matrices as

$$\tilde{\mathbf{H}}_i = \sqrt{\frac{N_t N_r}{\|\mathbf{H}_i\|_F^2}}\mathbf{H}_i, \tag{3}$$

$$\hat{\mathbf{H}}_i = \sqrt{\frac{N_s N_t N_r}{\sum_{i=1}^{N_s}\|\mathbf{H}_i\|_F^2}}\mathbf{H}_i. \tag{4}$$

It follows immediately that

$$\|\tilde{\mathbf{H}}_i\|_F^2 = N_t N_r, \tag{5}$$

$$\frac{1}{N_s}\sum_{i=1}^{N_s}\|\hat{\mathbf{H}}_i\|_F^2 = N_t N_r. \tag{6}$$

Thus in (3) every snapshot of the normalized channel matrix has a fixed Frobenius norm, whereas in (4) the time average of the Frobenius norm of the channel snapshots is fixed.

Applying the first normalization in (3), all snapshots of the channel produce the same SNR of $\frac{P}{{\sigma_n}^2}$. On the other hand using the second normalization of (4) preserves the relative difference in the channel power over time, but keeps the time average of the SNR at $\frac{P}{{\sigma_n}^2}$. This value of SNR is also equivalent to the average SNR of a Single-Input Single-Output (SISO) system with the same transmit and noise power. Therefore, these normalizations also allow a fair comparison of a MIMO system with its SISO counterpart.

III. MIMO MUTUAL INFORMATION

Assuming equal power transmission strategy, the mutual information between input and output of the channel is given by

$$I = \log_2\left|\mathbf{I} + \frac{P}{N_t{\sigma_n}^2}\mathbf{W}\right|, \tag{7}$$

where $\mathbf{I}$ is the identity matrix and the $K \times K$ matrix $\mathbf{W}$ is defined by

$$\mathbf{W} = \begin{cases} \mathbf{HH}^*, & N_t \geq N_r \\ \mathbf{H}^*\mathbf{H}. & N_t < N_r \end{cases} \tag{8}$$

In the high SNR regime, MI can be approximated as

$$I \simeq \log\left|\frac{P}{N_t{\sigma_n}^2}\mathbf{W}\right| = K\log\frac{P}{N_t{\sigma_n}^2} + \log|\mathbf{W}|. \tag{9}$$

Inserting (3) and (4) into (9) and using (8), corresponding MI's are given by

$$\begin{aligned} \tilde{I}_i &= K\log\frac{P}{N_t{\sigma_n}^2} + \log|\tilde{\mathbf{W}}| \\ &= K\log\frac{P}{N_t{\sigma_n}^2} + K\log\frac{N_t N_r}{\|\mathbf{H}_i\|_F^2} + \log|\mathbf{W}|, \end{aligned} \tag{10}$$

$$\begin{aligned} \hat{I}_i &= K\log\frac{P}{N_t{\sigma_n}^2} + \log|\hat{\mathbf{W}}| \\ &= K\log\frac{P}{N_t{\sigma_n}^2} + K\log\frac{N_s N_t N_r}{\sum_{i=1}^{N_s}\|\mathbf{H}_i\|_F^2} + \log|\mathbf{W}|. \end{aligned} \tag{11}$$

Therefore, the MI difference due to the application of the normalizations in (3) and (4) can be computed as

$$\Delta I = \hat{I}_i - \tilde{I}_i = K\log\frac{N_s\|\mathbf{H}_i\|_F^2}{\sum_{i=1}^{N_s}\|\mathbf{H}_i\|_F^2}. \tag{12}$$

IV. STATISTICAL ANALYSIS

In order to find the statistical properties of ΔI in (12), we focus on the i.i.d. Rayleigh fading channels, such that $[\mathbf{H}]_{ij} \sim \mathcal{CN}(0,1)$. Therefore $\|\mathbf{H}\|_F^2$ follows a Gamma distribution with parameters $N_t N_r$ and 1 denoted by $\Gamma(N_t N_r, 1)$.

Proposition 1: The probability density function (pdf) of ΔI in (12) is given by

$$f_{\Delta I}(w) = \frac{2^{wL}\ln 2}{K\mathrm{B}(a,b)}\frac{(N_s - 2^{w/K})^{b-1}}{N_s^{\,aN_s-1}}, \qquad (13)$$

where $\mathrm{B}(a,b)$ is the *beta function* defined by

$$\mathrm{B}(a,b) = \int_0^1 \frac{t^{a-1}}{(1+t)^{a+b}}dt.$$

Proof: Defining u, v and r as

$$u = \|\mathbf{H}_i\|_F^2 \;,\; v = \sum_{\substack{k=1\\k\neq i}}^{N_s}\|\mathbf{H}_k\|_F^2 \;,\; r = \frac{u}{u+v}, \qquad (14)$$

one can easily check that $u \sim \Gamma(N_t N_r, 1)$ and $v \sim \Gamma(N_t N_r(N_s-1), 1)$, hence r has a beta distribution with parameters, $a = N_t N_r = KL$ and $b = N_t N_r(N_s-1)$ denoted by $\mathcal{B}(a,b)$. The distribution of ΔI is followed by noting that ΔI can be written as

$$\Delta I = K\log(N_s r). \qquad (15)$$

■

Proposition 2: Mean and variance of ΔI in (12) is given by

$$\mathrm{E}[\Delta I] = \frac{K}{\ln 2}[\ln N_s + \Psi(a) - \Psi(a+b)], \qquad (16)$$

$$\mathrm{var}[\Delta I] = \left(\frac{K}{\ln 2}\right)^2[\Psi'(a) - \Psi'(a+b)]. \qquad (17)$$

where $\Psi(x)$ and $\Psi'(x)$ are *digamma* and *trigamma* functions given by $\Psi(x) = \frac{d}{dx}[\ln\Gamma(x)]$ and $\Psi'(x) = \frac{d^2}{dx^2}[\ln\Gamma(x)]$ with $\Gamma(x)$ being the *gamma* function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ [10].

Proof: The result immediately follows by noting that $\mathrm{E}[g(x)] = \int g(x) f_x(x)dx$ and using the integral derivations in the Appendix.

■

Fig. 1 plots the pdf of the MI difference given by (12) in a system with *K=5* and *L=8* for different values of N_s. Fig. 2 and Fig. 3 illustrate the effect of the number of antennas and snapshots on the mean and variance of the MI difference, respectively in which β denotes the ratio of the maximum and minimum number of antennas. Fixing the number of snapshots to $N_s = 100$, Fig. 4 and Fig. 5, respectively, show variations of the mean and variance of the MI difference as a function of the number of antennas. Discussion on the implications of the figures is deferred to the next section.

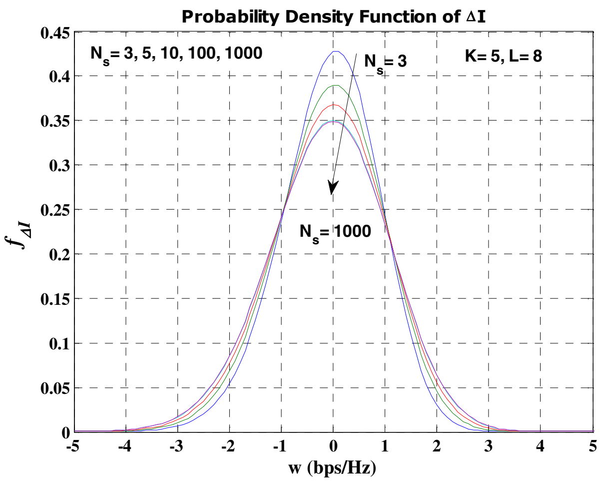

Fig. 1 Probability density function of ΔI for *K=5, L=8* and *Ns=3,5,10,100,1000.*

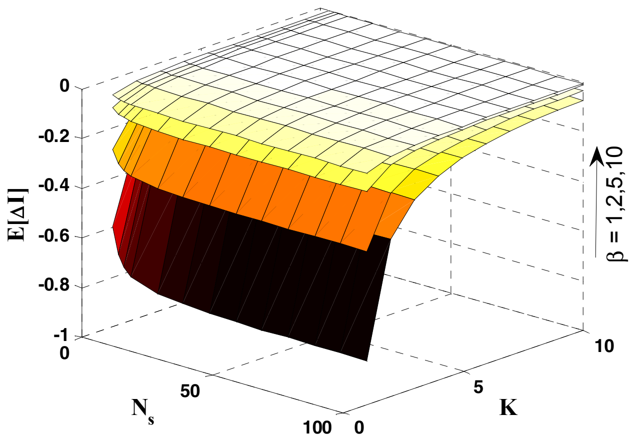

Fig. 2 Mean of ΔI against K and N_s for $\beta = 1,2,5,10$

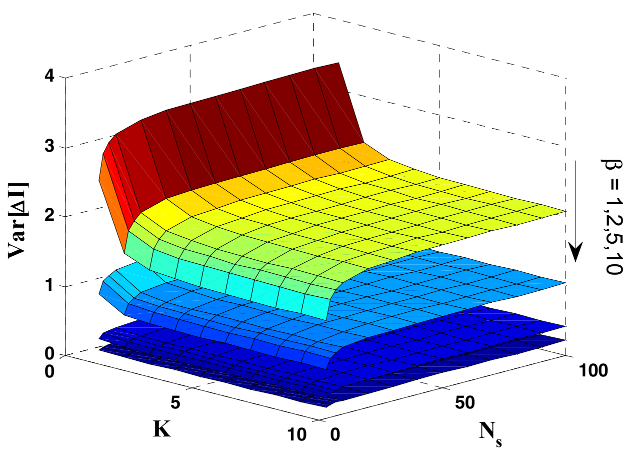

Fig. 3 Variance of ΔI against K and N_s for $\beta = 1,2,5,10$

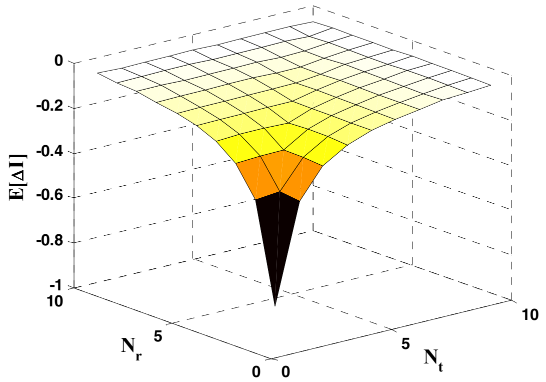

Fig. 4 Mean of ΔI as a function of the number of antennas for $N_s = 100$

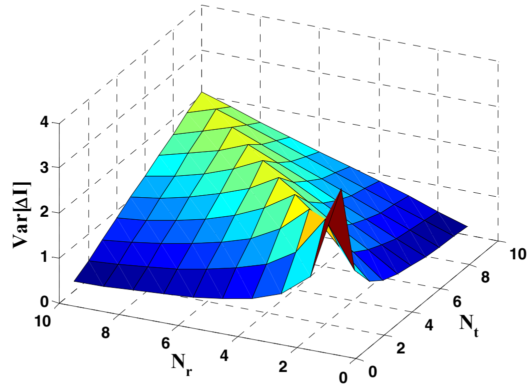

Fig. 5 Variance of ΔI as a function of the number of antennas for $N_s = 100$

V. DISCUSSION

In this section we focus on a qualitative discussion on the results derived in Section III and IV.

Remark 1: From (12), it is observed that the MI difference due to the two different normalization methods is determined by the instantaneous channel power transfer gain, i.e. the squared Frobenius norm of the channel matrix snapshots. In other words, the MI difference is not affected by the details of the eigenstructure of the channel as long as the Frobenius norms of the channel are not changed. It is worth noting that Salo *et al.* in [11] have proposed a decomposition of the mutual information in the high SNR regime into three independent terms, involving average SNR, channel fading and eigenvalue dispersion of the channel matrix. Interestingly, when the number of channel snapshots (N_s) becomes large enough such that the time average can be replaced by statistical mean in (12), ΔI coincides with the fading dependent term of the decomposition, which has been shown to be independent of a measure of the eigenvalue dispersion of the channel, referred to as "*ellipticity statistics.*"

Remark 2: As it can be seen from Fig. 1, the distribution of the MI difference reaches a statistical stability as N_s increases. The reason is that the average of the instantaneous squared Frobenius norms in (12) converges to their expectation and becomes independent of N_s . Accordingly, the mean and variance of the distribution stabilize with the increase of N_s, and the convergence occurs much faster for larger channel dimensions, as illustrated in Fig. 2 and Fig. 3.

Remark 3: Noting that $\mathrm{E}[\Delta I]=\mathrm{E}[\hat{I}_i]-\mathrm{E}[\tilde{I}_i]$, (16) gives the difference in the mean mutual information due to the two normalizations. The quantity $\mathrm{E}[I]$ represents the ergodic capacity of the underlying channel when the transmitter is uninformed, i.e. has no knowledge of channel state information (CSI). In order to enable the transmitter to control its power, we have implicitly assumed that the Frobenius norm of the channel is fed back to the transmitter. Such information may be exploited by the transmitter to achieve higher data rates. Therefore, we call $\mathrm{E}[I]$ simply the mean mutual information, though it still gives information about the behavior of the capacity.

Remark 4: From (16) one can observe that $\mathrm{E}[\Delta I]$ is always negative which is also evident from Fig. 2. Apparently, higher values of instantaneous SNR in case of slow power control cannot make up for the loss in MI incurred when the instantaneous SNR is below the average. The reason lies in the diminishing return of the MI, which is logarithmically dependent on the SNR in high SNR regime: the higher the SNR, the smaller the effect on capacity. Nevertheless the very small difference may sound counterintuitive as one may expect significant capacity loss due to imperfect power control. In fact, the amount of this loss is controlled by the extent of the variations of the instantaneous SNR; the higher is the variance of the SNR (i.e. more severe fading), the more the loss will be. This capacity loss for SISO channels in Rayleigh fading environment is 0.83 bps/Hz which can be seen in Fig. 2. However, as the number of antennas increases this loss becomes negligible. It is due to the inherent diversity of MIMO channels that causes a decrease in the fluctuations of the SNR.

Remark 5: From (17) , it can be concluded that for a fixed number of subchannels (i.e. constant $N_t \times N_r$), the absolute value of mean and variance of the difference in MI is maximized when the number of transmit and receive antennas are equal (i.e. $K = L$). This fact has been illustrated in Fig. 4 and Fig. 5. On the other hand, as the ratio of the number of antennas at the both ends increases, the difference in MI almost vanishes as shown in Fig. 2 and Fig. 3. It suggests that for highly asymmetrical systems, the instantaneous SNR becomes almost constant.

Remark 6: Although the results of Section IV are derived for temporally uncorrelated channels, they can be interpreted as an upper bound for temporally correlated channels. Since the former are more prone to radical changes in the channel transfer gain, the variance of the SNR is expected to be higher, and thus the mutual information of uncorrelated channels suffers more from imperfect power control. Nevertheless, the bound is expected to become tighter as the number of channel snapshots grows, since extending the averaging time horizon lets the channel experience most of its states during the averaging period. In such a case, it is the extent of the variations of the channel power- determined by the statistics of the channel- that counts, not the speed of variations- determined by the channel temporal correlation. On the other hand, spatial correlation degrades the diversity gain of the channel, and as such the results of this paper can be thought of as a lower bound for spatially correlated channels.

Remark 7: Following (12), it is easy to show that the loss in the sample mean of the MI due to imperfect power control, for channels of arbitrary statistics, is obtained via

$$\frac{1}{N_s}\sum_{i=1}^{N_s}\hat{I}_i-\tilde{I}_i = K\log(\frac{F_g}{F_a}), \quad (18)$$

where F_g and F_a are geometric and arithmetic means of the squared Frobenius norm of the channel snapshots respectively and are given by

$$F_a = \frac{1}{N_s}\sum_{i=1}^{N_s}\|\mathbf{H}_i\|_F^2, \quad (19)$$

$$F_g = \sqrt[N_s]{\prod_{i=1}^{N_s}\|\mathbf{H}_i\|_F^2}. \quad (20)$$

Again (18) underlines the impact of the temporal variations of the SNR on the capacity loss. Since $F_g \le F_a$, there is always a loss in sample mean of the mutual information due to imperfect power control, and this loss increases with the temporal variations of the SNR.

Remark 8: Finally, as opposed to the results of this work, the authors of [12] have reported a considerable capacity loss due to SNR temporal variations in a non-line-of-sight measurement. However, it is noted that, their conclusion is based on a measurement campaign in which the receiver is moving. Therefore, the path loss is changing from one snapshot to another. These different path losses cannot be removed from the measurements through normalization and therefore cause significant variations in the SNR, which in turn, result in high capacity loss. This effect has not been considered in the current work since a purely Rayleigh channel has been assumed. Nevertheless equation (18) that is derived with no pre-assumption about the underlying channel still should be able to predict the loss in the average MI.

VI. CONCLUSIONS

We presented analytical results on the distribution, mean and variance of the mutual information difference due to the application of two power normalization methods corresponding to slow and fast power control. We first derived our results for uncorrelated Rayleigh fading environments in high SNR regime, then discussed the extension of the results as upper bound and lower bound for temporally and spatially correlated channels, respectively. Furthermore, we provided a closed form expression for the sample mean of the mutual information difference for channels of arbitrary statistics. We related the mutual information difference to the extent of temporal fluctuations of the SNR and showed that for Rayleigh channels, this difference disappears for MIMO channels represented by large dimensional matrices, particularly when the channel matrix dimensions are highly asymmetric.

APPENDIX

Two integrals involved in the proof of Proposition 2 are

$$\frac{1}{\mathrm{B}(a,b)}\int_0^1 x^{a-1}(1-x)^{b-1}\ln x\,\mathrm{d}x = \Psi(a)-\Psi(a+b), \quad (21)$$

$$\frac{1}{\mathrm{B}(a,b)}\int_0^1 x^{a-1}(1-x)^{b-1}\ln^2 x\,\mathrm{d}x = \Psi'(a)-\Psi'(a+b)+[\Psi(a)-\Psi(a+b)]^2. \quad (22)$$

Proof: Differentiating the identity $\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, with respect to *a*, yields (21) and (22) follows by differentiating (21) with respect to *a* once more.

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